



Determination Premiums Motor Vehicle Insurance Using Bonus-Malus Optimal

Amalia Listiani ^{a,1,*}, Mitha Patricia ^{a,2}, Tiara Yulita ^{a,3}

^a Actuarial Science, Faculty of Science, Institut Teknologi Sumatera, Jl Terusan Ryacudu, Way Huwi, Jati Agung, Lampung Selatan, 35365, Indonesia

¹ amalia.listiani@at.itera.ac.id *; ² mitha.patricia3@gmail.com; ³ tiara.yulita@at.itera.ac.id

* Corresponding author

ARTICLE INFO

ABSTRACT

History

Submitted: September 10, 2025

Revised: October 7, 2025

Accepted: November 11, 2025

Keywords

Bayesian Method

Bonus Malus Optimal

Lognormal-Gamma

Motor Vehicle Insurance

Poisson-Lindley

The increasing number of motor vehicles in Sumatera has heightened accident risks, emphasizing the need for motor vehicle insurance to distribute risk between policyholders and insurers. Determining fair and risk-based premium requires consideration of each policyholder's claim history. This study aimed to determine motor vehicle insurance premiums using the optimal bonus-malus system based on claim data for the minibus category with comprehensive coverage in Sumatera during 2022. The proposed model extended the Bayesian bonus-malus framework by incorporating the trust region reflective (TRR) method for estimating claim severity and the Newton-Raphson method for estimating claim frequency, thereby enhancing parameter estimation accuracy and numerical stability. This approach offers a more equitable and precise premium adjustment mechanism aligned with individual risk levels, contributing to improved risk-based pricing, reduced underwriting losses, and greater transparency for policyholders. The results showed that the claim frequency followed the Poisson-Lindley distribution, while claim severity followed the lognormal-gamma distribution. Based on these models, the premium was computed by multiplying the basic premium by the relative value of the subsequent year and dividing it by the base relative value. Premium decrease in the absence of claims and increase when claims occur.

1. Introduction

The development of the automotive industry has increased the number of motor vehicles in Sumatera. According to data from Statistics Indonesia (Badan Pusat Statistik/BPS), the number of motor vehicles in Sumatra was around 32,86 units in 2023, ranking second in Indonesia after Java [1]. The growth in the number of vehicles increases various risks, such as accidents, damage, theft, and other hazards. Therefore, vehicle owners need to transfer these risks to insurance companies to reduce the financial impact arising from motor vehicle-related risks. In addition, insurance companies must also develop motor vehicle insurance products with accurate premium calculations.

Insurance is an agreement between an insurance company and a policyholder as the insured party, as stipulated in the policy [2]. The protection against losses provided serves as a risk mitigation measure against sudden adverse events. Motor vehicle insurance protects the insured vehicle as specified in the agreed policy. The existence of motor vehicle insurance is closely related to the increasing number of vehicles. The rapid growth of motor vehicles contributed to rising traffic and the number of traffic accidents [3]. There are two types of motor vehicle insurance coverage: total

loss only (TLO) and comprehensive [4]. Under TLO coverage, compensation is paid when the reported vehicle damage exceeds 75% of the insured value. Under comprehensive coverage, compensation is paid for all reported losses, provided the amount exceeds the deductible, which is the portion of loss borne by the insured [5].

Compensation, or claims, for losses suffered by the insured, such as in the event of an accident will be provided by the insurer if the insured has fulfilled their obligation to pay premiums [6]. The fundamental principle of insurance risk management is the equitable distribution of losses, where the financial burden of the few is shared by the many, achieved through the pooling of premiums [7]. A premium is a payment that must be made by the insured to the insurer, the amount of which depends on the vehicle's age, the insured value, and other factors [8]. If the premium set by the insurer is higher than it should be to gain more profit, fewer customers will purchase the product. Conversely, if the premium is set lower than it should be, the insurer will earn less profit or may even incur losses due to an inability to cover policyholders' claims. Therefore, optimal premium calculation is essential.

One system that can be used to determine motor vehicle insurance premiums is the bonus–malus system. This system considers historical claim data for both claim frequency and claim severity to determine whether the policyholder will receive a bonus (premium discount) or a malus (premium surcharge) [9]. A bonus is granted to policyholders with a good claim's history, meaning they have made no claims. Policyholders with a poor claim's history, or those who have made one or more claims, will receive a malus to offset past claim payments [10]. The bonus–malus system is divided into two types: the classical bonus–malus, which relies solely on claim frequency data, and the optimal bonus–malus, which considers both claim frequency and severity. The premium amount in the bonus–malus system is determined using the expected value principle [11].

The previous study demonstrated that the optimal bonus–malus system (OBMS) offers a more accurate and fair premium adjustment mechanism than the classical bonus–malus system (CBMS) [10]. Their approach integrated both claim frequency and claim severity, enabling insurers to better capture the risk profile of each policyholder. However, their model relied on parameter estimation techniques that assumed relatively simple claim distributions and linear relationships, limiting its flexibility when applied to more complex or heavy-tailed insurance data. In contrast, previous study incorporated the Poisson–Lindley distribution for claim frequency and the Lognormal–Gamma distribution for claim severity, reflecting a more realistic and flexible modeling framework [10]. Nevertheless, their parameter estimation employed the Newton–Raphson iterative method, which, while effective for smooth functions, is sensitive to initial values and may converge to local minima when the likelihood surface is highly nonlinear.

To address this limitation, the present study extends model by adopting the optimal bonus–malus framework, while enhancing the parameter estimation process through the trust region reflective (TRR) method. The TRR approach provides improved numerical stability and convergence reliability compared to the Newton–Raphson method, particularly for nonlinear optimization problems with multiple parameters. This methodological integration allows the proposed model to achieve higher parameter estimation accuracy and greater actuarial fairness. For insurers, this contributes to a more precise risk-based pricing strategy, potentially reducing underwriting losses and enhancing portfolio stability. Meanwhile, for policyholders, this model promotes fairness and transparency in premium determination, ensuring that everyone's premium level more accurately represents their historical claim behavior.

Accordingly, this study aimed to determine motor vehicle insurance premiums using the optimal bonus–malus system, based on 2022 claim data for buses and minibuses in Sumatera with comprehensive coverage. The expected output of this study is a model for calculating motor vehicle insurance premiums using the determined claim severity and claim frequency distributions, along with an example of premium calculation for a motor vehicle insurance company applying the optimal bonus–malus system with the expected value principle.

2. Method

This study employed a quantitative approach, developing a mathematical model to determine motor vehicle insurance premiums. The calculation of the premium amount requires prior identification of the probability distribution of the claim data. The dataset utilized in this study comprised claim frequency and claim severity records for motor vehicles classified under the minibus category in Sumatera, with comprehensive coverage, for the year 2022. The data were obtained from an insurance company operating in Indonesia. For confidentiality and data protection reasons, the identity of the company cannot be disclosed.

2.1. Chi-Squared Test of Independence

The chi-square test of independence is a statistical test used to determine whether there is an association between two random variables or whether they are independent. The hypotheses are defined as H_0 : claim frequency and claim severity are independent, and H_1 claim frequency and claim severity are not independent. The decision rule is to reject H_0 when $X^2_{calculated} \geq X^2_{table}$ where X^2_{table} is determined based on the degrees of freedom and the significance level, while $X^2_{calculated}$ is obtained using (1) [12].

$$X^2_{calculated} = \sum_j \sum_k \frac{(E_{jk} - O_{jk})^2}{E_{jk}} \quad (1)$$

where E_{jk} is the expected value for the j th row and k th column obtained from the following equation $E_{jk} = \frac{m_k \times n_j}{N}$. The values of E_{jk} and O_{jk} are obtained from the contingency table [11]. The observation value in the j th row and k th column is notated with O_{jk} , then n_j is the sum of the observation values in the j th row, m_k is the sum of the observation values in the k th column, and N is the total observation value.

2.2. Poisson-Lindley Distribution

In motor insurance, each policyholder possesses a distinct level of underlying risk associated with the likelihood of an accident. This level of risk, referred to as the risk parameter, is considered a random variable that varies across individuals. The distribution of this parameter is known as the prior distribution. Combining the Poisson distribution with a prior distribution produces heavier tails than the standard Poisson distribution, thereby offering a more accurate representation of claim frequency data. Therefore, this study proposed modelling the claim frequency using a Poisson–Lindley mixture distribution.

In this study, the claim data were assumed to follow a Poisson distribution with parameter K , and the K parameters followed a Lindley distribution with parameter δ . K was distributed according to the Poisson conditional on the parameter $\Theta = \theta$. The Poisson distribution is a discrete probability distribution used to determine the probability of an event occurring within a specific period, with the probability mass function given as (2) [13]

$$p(k|\theta) = \frac{(e^{-\theta}\theta^k)}{k!}; k = 0,1,2, \dots, \text{ and } \theta > 0. \quad (2)$$

The expected value of Poisson random variable was given by $E[K|\theta] = \theta$. Each policyholder was assumed to have inherent expected risk level, represented by θ , corresponding to the average number of claims associated with that individual.

Meanwhile, the Lindley distribution is a probability distribution for lifetime data and can address the problem of overdispersion, a condition in which the variance exceeds the mean. θ was distributed according to the Lindley distribution with parameter δ . The probability density function of the Lindley distribution is as follows [14].

$$\pi(\theta) = \frac{\delta^2}{\delta+1} (1 + \theta)e^{-\delta\theta}, \theta > 0, \delta > 0. \quad (3)$$

The Poisson–Lindley distribution is a compound distribution combining the discrete Poisson distribution with the continuous Lindley distribution, with parameter δ . The mixed Poisson with Lindley is:

$$p(k) = \sum_{n=0}^{\infty} p(k|\theta)\pi(\theta).$$

If a random variable K follows a Poisson-Lindley distribution, it can be denoted as $K \sim \text{Poisson-Lindley}(\delta)$ with its probability mass function given in (4).

$$p(k) = \frac{\delta^2(k+\delta+2)}{(1+\delta)^{k+3}}, \tag{4}$$

where $k = 0, 1, 2, \dots$ and $\delta > 0$.

2.3. Lognormal-Gamma Distribution

In insurance portfolios, heavy-tailed distributions such as gamma and lognormal are frequently employed to model claim severity. When these claim severity distributions are combined with a prior distribution, the resulting mixture tends to produce heavier tails, thereby offering a better fit to empirical claim severity data. In this study, the claims data were assumed to follow a lognormal distribution with parameter (μ, λ) , with $\lambda = \frac{1}{\sigma^2}$, and λ is a gamma distribution with parameters (α, β) .

Let $X|\Lambda$ be a random variable of the claim size of each insured person and assume that $X|\Lambda$ follows a Lognormal distribution with given parameter $\lambda = \frac{1}{\sigma^2}$. The Lognormal distribution is one of the distributions suitable for claim severity data and has two parameters, namely μ and σ [15].

$$f(x|\lambda) = \frac{\sqrt{\lambda}}{x\sqrt{2\pi}} e^{-\frac{\lambda(\ln x - \mu)^2}{2\sigma^2}}, \quad x > 0, \mu > 0, \lambda > 0. \tag{5}$$

The parameter λ varies across policyholders; therefore, it is reasonable to model λ as a random variable following a probability distribution. The gamma prior distribution for λ , with parameters given as α and β , its probability density function [15] is given by

$$\pi(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda}, \quad \lambda > 0, \alpha > 0, \beta > 0. \tag{6}$$

The gamma function $\Gamma(\alpha)$ can be expressed in factorial form as:

$$\Gamma(\alpha) = \int_0^{\infty} e^{-x} x^{\alpha-1} dx = (\alpha - 1)!.$$

The lognormal–gamma distribution is a compound distribution combining the continuous lognormal distribution with the continuous gamma distribution, with three parameters: α, β , and μ . If a random variable X follows a lognormal–gamma distribution, it can be denoted as $X \sim \text{lognormal-gamma}(\alpha, \beta, \mu)$ with its probability density function given in (7).

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)x\sqrt{2\pi}} \frac{\Gamma(\alpha + \frac{1}{2})}{\left(\frac{(\ln x - \mu)^2}{2} + \beta\right)^{\alpha + \frac{1}{2}}}. \tag{7}$$

2.4. Maximum Likelihood Estimation

The maximum likelihood estimation (MLE) method is a parameter estimation technique for a probability distribution that works by maximizing the likelihood function. The step for determining the parameter estimates in (8)–(10) [15] are as follows.

a. Formulate the Likelihood function

$$L(\theta) = \prod_{i=1}^n f_{X_i|\theta}(x_i | \theta). \tag{8}$$

b. Transforming the Likelihood function into its natural logarithm form:

$$l(\theta) = \ln\left(\prod_{i=1}^n f_{X_i|\theta}(x_i | \theta)\right) = \sum_{i=1}^n \ln(f_{X_i|\theta}(x_i | \theta)). \quad (9)$$

c. Differentiating with respect to the parameters and setting the derivatives equal to zero:

$$\frac{d}{d\theta} l(\theta) = \frac{d}{d\theta} \left(\sum_{i=1}^n \ln(f_{X_i|\theta}(x_i | \theta))\right) = 0. \quad (10)$$

2.5. Newton Raphson Method

The Newton–Raphson method is a numerical method used to find the roots of a nonlinear equation [16]. This method applies the Taylor series expansion (11):

$$f(x) = f(x_0) + \frac{(x-x_0)f'(x_0)}{1!} + \frac{(x-x_0)^2 f''(x_0)}{2!} + \dots + \frac{(x-x_0)^n f^n(x_0)}{n!}. \quad (11)$$

By applying the Taylor series and truncating after the second term, the iterative formula to obtain x for $f(x) = 0$ is (12).

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \quad (12)$$

The iteration process continues until the relative change ratio between two successive values is less than or equal to a predetermined error tolerance, which must be very small but greater than zero [17].

2.6. Trust Region Reflective (TRR) Method

The trust region reflective (TRR) method is a parameter estimation approach for nonlinear least squares (NLS) problems, which optimizes a quadratic approximation of a nonlinear objective function within a specific trust region that serves as an appropriate size constraint [18]. The general form of the NLS (13):

$$Y_i = f(\theta, x_i) + Z_i. \quad (13)$$

The general form of the TRR method (14).

$$\min_s q(s) = \min_s \left(\frac{1}{2} s^T \mathbf{B}_k s + \nabla f(x^k)^T s \right), \quad (14)$$

where

$$\mathbf{B}_k \stackrel{\text{def}}{=} \mathbf{H}_k + \mathbf{D}_k^{-1} \text{diag} \left(\nabla f(x^k) \right) \mathbf{J}_x(x^k) \mathbf{D}_k^{-1}. \quad (15)$$

$$\mathbf{H}_k \stackrel{\text{def}}{=} \nabla^2 f(x^k). \quad (16)$$

$$\mathbf{D}_k \stackrel{\text{def}}{=} \text{diag} \left(\sqrt{|v(x^k)|} \right). \quad (17)$$

$$\mathbf{J}_x(x^k) \stackrel{\text{def}}{=} \left(\nabla |v_1|^T \ : \ \nabla |v_n|^T \right). \quad (18)$$

Based on (15)–(18), \mathbf{B}_k is the approximated Hessian matrix, \mathbf{H}_k is the Hessian matrix, \mathbf{D}_k is a diagonal matrix, v denotes the residual values, and \mathbf{J}_x is the Jacobian matrix. The length of the vector $\mathbf{D}_k^{-1} s$ must be less than or equal to the trust-region radius, denoted as $\|\mathbf{D}_k^{-1} s\| \leq \Delta_k$ where $\Delta_k > 0$. The vector $\mathbf{v}(x) = (v_1(x), \dots, v_n(x))^T$ is defined based on (19)–(22) as follows:

$$v_i \stackrel{\text{def}}{=} x_i - ub_i \text{ if } \nabla(f(x))_i < 0 \text{ and } ub_i < \infty, \quad (19)$$

$$v_i \stackrel{\text{def}}{=} x_i - lb_i \text{ if } \nabla(f(x))_i \geq 0 \text{ and } lb_i < \infty, \quad (20)$$

$$v_i \stackrel{\text{def}}{=} -1 \text{ if } \nabla(f(x))_i < 0 \text{ and } ub_i = \infty, \quad (21)$$

$$v_i \stackrel{\text{def}}{=} 1 \text{ if } \nabla(f(x))_i \geq 0 \text{ and } lb_i = -\infty. \quad (22)$$

2.7. Chi-Squared Goodness of Fit Test

The chi-squared test is a goodness-of-fit test commonly used for discrete distributions to verify whether the data fits the obtained distribution. The hypotheses used are as follows:

H_0 : The data follows a certain distribution.

H_1 : The data follows another distribution.

The above hypotheses will reject H_0 when $X_{calculated}^2 \geq X_{table}^2$ where X_{table}^2 is determined from the degrees of freedom and the significance level, while $X_{calculated}^2$ is computed using (23) [19]:

$$X_{calculated}^2 = \sum_{j=1}^k \frac{(E_j - O_j)^2}{E_j}, \tag{23}$$

where k is the value for the number of claims, with O_j being the value of the j th observation and E_j being the expected value at j .

2.8. Anderson-Darling Goodness of Fit Test

The Anderson–Darling test is a goodness-of-fit test commonly used for continuous distributions to verify whether the data fits the obtained distribution [20]. The hypotheses used are as follows:

H_0 : The data follows a certain distribution.

H_1 : The data follows another distribution.

The above hypotheses will reject H_0 when the test statistic (A^2) > exceeds the critical value determined from the significance level, as given in Table 1.

Table 1. Anderson-Darling Test Critical Values

Significance Level	0.01	0.025	0.05	0.10	0.15
Critical Values	3.853	3.070	2.492	1.933	1.610

Meanwhile A^2 is calculated using (24).

$$A^2 = -n - \frac{1}{n} \sum_{i=1}^n (2i - 1) [\ln (F(x_i)) + \ln (1 - F(x_{n-i+1}))], \tag{24}$$

where A^2 is the value of the Anderson-Darling test statistic, n is the number of data points, and $F(x_i)$ is the cumulative function for data x_i .

2.9. Bonus-Malus System

The bonus–malus system is a method of determining the amount of insurance premium that must be paid by the policyholder based on their claims history. If the policyholder does not file any claims, the company will grant a bonus in the form of a premium reduction. Conversely, policyholders who frequently file claims will receive a malus or premium increase. The bonus–malus system is divided into two types: the CBMS, which only considers the number of claims, and the OBMS, which takes into account both the number of claims and the claim amounts of policyholders. In premium calculation, the first step is to determine the relative value (NR) obtained using *the* expected value principle. In the CBMS, the relative value $n + 1$ denoted as NR_{n+1} is obtained from the expected value of the number of claims (25) [21]:

$$NR_{n+1} = E[\textit{Posterior distribution for the number of claims}]. \tag{25}$$

In the OBMS, NR_{n+1} is obtained by multiplying the expected value of the posterior distribution of the number of claims with the expected value of the claim size, as given in (26):

$$NR_{n+1} = E[\textit{Posterior of claim frequency}] \times E[\textit{Posterior of claim severity}]. \tag{26}$$

The relative value obtained is then used in the comparative equation to determine the premium amount for the following n years (27).

$$\frac{NR_1}{NR_{n+1}} = \frac{\textit{Premium}_1}{\textit{Premium}_{n+1}}. \tag{27}$$

$$Premium_{n+1} = \frac{Premium_1 \times NR_{n+1}}{NR_1}.$$

3. Results and Discussion

3.1. Descriptive Statistics

The data used were motor vehicle insurance claim data for the minibus category in Sumatra with comprehensive coverage for the year 2022. From the data used, total number of policies were 877, consisting of 546 policies with no claims and 331 policies with claims. The descriptive statistics for the claim amount and claim frequency for the 331 policies with claims can be seen in Table 2. The claim frequency had a minimum value of 1, a median of 1, and a maximum of 3. The mean claim frequency was 1, meaning that most policyholders made one claim. The standard deviation of 0.4470 indicated that the data were not spread far from the mean. The total number of claims from the 331 policies were 408 claims. For the claim amount data, the minimum value was IDR300,000, the median value was IDR19,043,520, and the maximum claim amount was IDR 341,860,800. The mean claim amount was IDR38,416,213, representing the average amount paid by the company, which is relatively high. The standard deviation was IDR50,276,513, indicating that the data were widely spread from the mean. The total amount of claims paid by the company was IDR 12,715,766,397.

Table 2. Summary Statistics of Claim Data

Indicator	Claim Frequency	Claim Amount (IDR)
Minimum	1	300,000
Median	1	19,043,520
Maximum	3	341,860,800
Mean	1	38,416,213
Standard Deviation	0.4470	50,276,513
Total	408	12,715,766,397

3.2. Chi-Squared Test of Independence

One of the requirements for this study is that the claim frequency data and claim amount data must be independent, hence a chi-squared test of Independence was conducted with the following hypotheses

H_0 : Claim frequency and claim amount are independent.

H_1 : Claim frequency and claim amount are not independent.

The significance level used was 5% with 634 degrees of freedom, and the critical region was to reject H_0 if $X^2_{calculated} \geq X^2_{table}$. Based on (1), the value of X^2_{table} was 693,687, while $X^2_{calculated}$ was 655,310. Therefore, the conclusion was that claim frequency and claim amount are independent, so the next steps in determining vehicle insurance premiums using the bonus-malus system could be carried out.

3.3. Determining the Model Distribution

Determining the model distribution was conducted after the independence test between claim frequency and claim amount. The probability density function for each variable's distribution was obtained and subsequently used to determine the equations for calculating vehicle insurance premiums. The distribution of claim frequency was discrete, while the claim amount distribution was continuous. Assume that the number of claims k is distributed according to the Poisson conditional on the parameter θ , where θ follows a Lindley distribution with parameter δ . This combination forms a Poisson-Lindley distribution as shown in (4). In this study, the claim amount (X) was assumed to follow a lognormal distribution with parameters (μ, λ) , where $\lambda = \frac{1}{\sigma^2}$, and λ follows a

gamma distribution with parameter (α, β) . This combination forms a lognormal-gamma distribution as shown in (7).

3.4. Estimation of Parameter

Parameter estimation in a distribution is crucial in this study, as the estimated parameters of the claim frequency and claim amount distributions was used for model goodness-of-fit testing, as well as for calculating premiums. The primary method used was MLE, which might be followed by other methods if a closed-form expression could be derived.

3.4.1. Parameter Estimation of Claim Frequency

Claim frequency data denoted with k was assumed to follow a Poisson-Lindley distribution with a probability function as shown in (4) with parameter δ . The value of δ could be estimated using the MLE method with the following steps:

- a. Form the likelihood function based on the probability function of the Poisson-Lindley distribution in (4):

$$\begin{aligned} L(\delta; k_i) &= \prod_{i=1}^n p(k_i) = \prod_{i=1}^n \frac{\delta^2(k_i + \delta + 2)}{(1 + \delta)^{k_i+3}} \\ &= \left(\frac{\delta^2(k_1 + \delta + 2)}{(1 + \delta)^{k_1+3}} \right) \left(\frac{\delta^2(k_2 + \delta + 2)}{(1 + \delta)^{k_2+3}} \right) \dots \left(\frac{\delta^2(k_n + \delta + 2)}{(1 + \delta)^{k_n+3}} \right) \\ &= \frac{\delta^{2n} \prod_{i=1}^n (k_i + \delta + 2)}{(1 + \delta)^{\sum_{i=1}^n k_i + 3n}} \end{aligned} \tag{28}$$

- b. Transform (28) into the natural logarithm (ln) form, resulting in (29):

$$\begin{aligned} l(\delta; k_i) &= \ln \left(\frac{\delta^{2n} \prod_{i=1}^n (k_i + \delta + 2)}{(1 + \delta)^{\sum_{i=1}^n k_i + 3n}} \right) \\ &= \ln \delta^{2n} + \ln \left(\prod_{i=1}^n (k_i + \delta + 2) \right) - \ln ((1 + \delta)^{\sum_{i=1}^n k_i + 3n}) \\ &= 2n \ln \delta + \sum_{i=1}^n \ln (k_i + \delta + 2) - \ln(1 + \delta) \sum_{i=1}^n (k_i + 3). \end{aligned} \tag{29}$$

- c. The result of the transformation in (29) is then differentiated with respect to the parameter δ and set equal to 0:

$$\begin{aligned} \frac{d}{d\delta} l(\delta; k_i) &= 0 \\ 0 &= \frac{d}{d\delta} \left(2n \ln \delta + \sum_{i=1}^n \ln (k_i + \delta + 2) - \ln \left((1 + \delta) \sum_{i=1}^n (k_i + 3) \right) \right) \\ 0 &= \frac{2n}{\delta} + \sum_{i=1}^n \frac{1}{(k_i + \delta + 2)} - \frac{1}{(1 + \delta)} \sum_{i=1}^n (k_i + 3). \end{aligned} \tag{30}$$

Based on (30), the parameter δ cannot be directly estimated because the equation is in closed form. Therefore, a numerical approach was needed to obtain the final result, namely the Newton-Raphson iteration, with the equation form given in (12). The computational estimation of the parameter δ yielded a value of 2.7106 after seven iterations until convergence.

3.4.2. Parameter Estimation of Claim Severity

The claim frequency data were assumed to follow a lognormal-gamma distribution, with the probability function given in (7) and parameters α , β , and μ . The values of parameters α , β , and μ can be estimated using the MLE method through the following steps:

- a. Formulate the likelihood function based on the probability function of the lognormal-gamma distribution in (7).

$$L(\alpha, \beta, \mu ; x_i) = \prod_{i=1}^n f(x_i) = \prod_{i=1}^n \frac{\beta^\alpha}{\Gamma(\alpha) x_i \sqrt{2\pi}} \frac{\Gamma\left(\alpha + \frac{1}{2}\right)}{\left(\frac{(\ln x_i - \mu)^2}{2} + \beta\right)^{\alpha + \frac{1}{2}}}$$

$$= \left(\frac{\beta^{\alpha n}}{\Gamma(\alpha)^n \prod_{i=1}^n x_i (\sqrt{2\pi})^n} \frac{\Gamma\left(\alpha + \frac{1}{2}\right)^n}{\prod_{i=1}^n \left(\frac{(\ln x_i - \mu)^2}{2} + \beta\right)^{\alpha + \frac{1}{2}}} \right). \quad (31)$$

b. Transform (31) into the natural logarithm (\ln) form.

$$l(\alpha, \beta, \mu ; x_i) = \ln \left(\frac{\beta^{\alpha n}}{\Gamma(\alpha)^n \prod_{i=1}^n x_i (\sqrt{2\pi})^n} \frac{\Gamma\left(\alpha + \frac{1}{2}\right)^n}{\prod_{i=1}^n \left(\frac{(\ln x_i - \mu)^2}{2} + \beta\right)^{\alpha + \frac{1}{2}}} \right)$$

$$= n \left(\alpha \ln(\beta) + \ln \Gamma\left(\alpha + \frac{1}{2}\right) - \ln \Gamma(\alpha) - \ln(\sqrt{2\pi}) \right) - \sum_{i=1}^n \ln x_i$$

$$- \left(\alpha + \frac{1}{2}\right) \sum_{i=1}^n \ln \left(\frac{(\ln x_i - \mu)^2}{2} + \beta \right) \quad (32)$$

c. The transformed result in (32) was differentiated with respect to each parameter, and its value was set to 0, with Parameter α

$$\frac{d}{d\alpha} l(\alpha, \beta, \mu ; x_i) = 0$$

$$0 = n \left(\ln(\beta) + \frac{d}{d\alpha} \ln \Gamma\left(\alpha + \frac{1}{2}\right) - \frac{d}{d\alpha} \ln \Gamma(\alpha) \right) - \sum_{i=1}^n \ln \left(\frac{(\ln x_i - \mu)^2}{2} + \beta \right). \quad (33)$$

Meanwhile, Parameter β is as follows.

$$\frac{d}{d\alpha} l \frac{d}{d\beta} l(\alpha, \beta, \mu ; x_i) = 0$$

$$0 = \frac{\alpha n}{\beta} - \left(\alpha + \frac{1}{2}\right) \sum_{i=1}^n \frac{1}{\left(\frac{(\ln x_i - \mu)^2}{2} + \beta\right)} \quad (34)$$

Parameter μ is as follows.

$$\frac{d}{d\mu} l(\alpha, \beta, \mu ; x_i) = 0$$

$$0 = \left(\alpha + \frac{1}{2}\right) \sum_{i=1}^n \left(\frac{(\ln x_i - \mu)}{\left(\frac{(\ln x_i - \mu)^2}{2} + \beta\right)} \right). \quad (35)$$

Based on (33), (34), and (35), the three parameters of the lognormal-gamma distribution cannot be estimated directly because these equations are in a closed form. Therefore, an advanced method was required to obtain the parameter estimates, namely the NLS method with TRR as shown in (10).

The estimation results of the lognormal-gamma distribution parameters obtained through computational processes after 293 iterations are presented in Table 3.

Table 3. Parameter Estimation Results for Claim Size

Parameter	Value
α	0.2999
β	4.4377
μ	10.5574

3.5. Model Goodness-of-Fit Test

A model goodness-of-fit test is conducted to ensure that the available data matches the distribution being used. This test can be performed once there is information regarding the chosen distribution, its probability function, and the estimated parameter values. In this case, the goodness-of-fit tests used were the chi-squared test for claim frequency data and the Anderson-Darling test for claim size data. The use of these two tests was tailored to the nature of the data. The chi-square test was applied to claim frequency data, which are discrete, as they effectively assess the fit between the theoretical distribution and the observed data. Meanwhile, the Anderson–Darling test was used for claim severity data, which are continuous, because they are more sensitive to differences in the tails of the distribution, making them suitable for insurance claim data that often exhibit extreme values. This selection ensures that the goodness-of-fit results are both accurate and representative.

3.5.1. Chi-Squared Test

The chi-squared test was used to verify whether the claim frequency data followed the Poisson-Lindley distribution, using the probability function and the parameter estimates obtained earlier. The probability function for the Poisson-Lindley distribution can be found in (13) and (14), with the estimated parameter value $\hat{\delta} = 2.7106$. The hypotheses are H_0 whether the data follows a Poisson-Lindley distribution and H_1 if the data follows another distribution.

Table 4. Calculation of $X^2_{calculated}$

Claim Count	Number of Policy	Observed Value	Expected Value	Chi- Squared Calculated
0	546	546	594,12047	3.89749
1	262	808	788,22486	0.49612
2	61	869	849,69586	0.43857
3	8	877	868,73084	0.07871
Total	877			4.91089

Using a 5% significance level and 2 degrees of freedom, the critical value was $X^2_{table} = 5.99$, while the calculated value $X^2_{calculated}$ was presented in Table 4. Based on the calculation, the value obtained was $X^2_{calculated} = 4.91089$, since $X^2_{calculated} (4.91089) < X^2_{table} (5.99)$, H_0 was not rejected. Therefore, the claim frequency data followed the Poisson-Lindley distribution.

3.5.2. Anderson-Darling Test

The Anderson-Darling test was used to determine whether the claim size data followed the lognormal-gamma distribution, using the probability function and parameter estimates obtained earlier. The probability function for the lognormal-gamma distribution can be found in (7), with estimated parameters $\hat{\alpha} = 0.29999$, $\hat{\beta} = 4.4277$, and $\hat{\mu} = 10.5574$. The hypotheses are as follows:

H_0 : The data follows a lognormal-gamma distribution

H_1 : The data follows another distribution.

Using a 5% significance level, the critical value was 2.492, while the computed test statistic was $A^2 = 0.71147$. since $A^2 (0.71147) < \text{critical value} (2.492)$, H_0 was not rejected. Thus, the claim size data followed the lognormal-gamma distribution.

3.6. Posterior Distribution of λ (Claim Frequency Rate)

For claim frequency data, the model used was the Poisson distribution with the probability function shown in (2). The Poisson probability function forms a likelihood function as in (8), which becomes:

$$L(\theta; k_i) = p_{K|\theta}(k|\theta) = \prod_{i=1}^n p(k_i|\theta) = \prod_{i=1}^n \frac{e^{-\theta} \theta^{k_i}}{k_i!} = \frac{e^{-\theta n} \theta^{\sum_{i=1}^n k_i}}{\prod_{i=1}^n k_i!}. \quad (36)$$

In (36), if variable θ is isolated, other variables can be ignored. Assuming $\sum_{i=1}^n k_i = M$, (36) can be simplified to:

$$p_{K|\theta}(k|\theta) \propto e^{-\theta n} \theta^M. \quad (37)$$

The prior distribution for the claim frequency data in this study followed the Lindley distribution, whose probability function is shown in (3). By ignoring variables other than θ because constants do not depend on θ (such as $\frac{\delta^2}{1+\delta}$), it becomes:

$$\pi(\theta) \propto (\theta + 1)e^{-\delta\theta}. \quad (38)$$

Using (37) and (38), the posterior distribution function can be expressed as:

$$\begin{aligned} \pi_{\theta|K}(\theta|k) &\propto \frac{e^{-\theta n} \theta^M (\theta + 1) e^{-\delta\theta}}{\int_0^\infty e^{-\theta n} \theta^M (\theta + 1) e^{-\delta\theta} d\theta} \\ &\propto \frac{(n + \delta)^{M+2} e^{-\theta(n+\delta)} (\theta^M) (\theta + 1)}{M! (M + 1 + n + \delta)} \end{aligned} \quad (39)$$

The posterior distribution was then used to determine its expected value, as follows:

$$\begin{aligned} E(\theta|k) &= \int_0^\infty \theta (\pi_{\theta|K}(\theta|k)) d\theta \\ &= \int_0^\infty \theta \frac{(n + \delta)^{M+2} e^{-\theta(n+\delta)} (\theta^M) (\theta + 1)}{M! (M + 1 + n + \delta)} d\theta = \frac{(M + 1)(M + 2 + n + \delta)}{(n + \delta)(M + 1 + n + \delta)} \end{aligned} \quad (40)$$

3.7. Posterior Distribution for Claim Size Data

In this study, the model used for claim size data was the lognormal distribution with the probability function given in (5). This probability function was then used to form the likelihood function as follows:

$$L(\lambda; x_k) = f_{X|\lambda}(\lambda) = \prod_{i=1}^n p(k_i|\lambda) = \prod_{i=1}^n \frac{e^{-\lambda \lambda^{k_i}}}{k_i!} = \frac{\lambda^{\frac{M}{2}}}{(\prod_{k=1}^M x_k) (2\pi)^{\frac{M}{2}}} e^{-\frac{\lambda}{2} \sum_{k=1}^M (\ln x_k - \mu)^2}. \quad (41)$$

In (41), if variable λ , is isolated, other variables can be ignored, where x_k is the claim size for claim number k with $k = 0, 1, 2, \dots, M$. Therefore, (41) can be simplified to:

$$f_{X|\lambda}(x|\lambda) \propto \lambda^{\frac{M}{2}} e^{-\frac{\lambda}{2} \sum_{k=1}^M (\ln x_k - \mu)^2}. \quad (42)$$

The prior distribution used for the large claim size data in this study was the gamma distribution, with the probability density function shown in (6). The probability density function of the gamma distribution can be simplified by ignoring variables other than the variable λ , resulting in (43).

$$\pi(\lambda) \propto \lambda^{\alpha-1} e^{-\lambda\beta} \tag{43}$$

Using (42) and (43), the posterior distribution function can be formulated as follows:

$$\begin{aligned} \pi_{\lambda|x}(\lambda|x) &\propto \frac{\lambda^{\frac{M}{2}} e^{-\frac{\lambda}{2} \sum_{k=1}^M (\ln x_k - \mu)^2} \lambda^{\alpha-1} e^{-\lambda\beta}}{\int_0^\infty \lambda^{\frac{M}{2}} e^{-\frac{\lambda}{2} \sum_{k=1}^M (\ln x_k - \mu)^2} \lambda^{\alpha-1} e^{-\lambda\beta} d\lambda} \\ &\propto \frac{\left(\frac{1}{2} \sum_{k=1}^M (\ln x_k - \mu)^2 + \beta\right)^{\frac{M}{2} + \alpha}}{\Gamma\left(\frac{M}{2} + \alpha\right)} \lambda^{\frac{M}{2} + \alpha - 1} e^{-\lambda\left(\frac{1}{2} \sum_{k=1}^M (\ln x_k - \mu)^2 + \beta\right)}. \end{aligned} \tag{44}$$

The posterior distribution function above was then used to determine its expected value, calculated as follows:

$$\begin{aligned} E(\lambda|x) &= \int_0^\infty \lambda \pi_{\lambda|x}(\lambda|x) d\lambda \\ &= \int_0^\infty \lambda \frac{\left(\frac{1}{2} \sum_{k=1}^M (\ln x_k - \mu)^2 + \beta\right)^{\frac{M}{2} + \alpha}}{\Gamma\left(\frac{M}{2} + \alpha\right)} \lambda^{\frac{M}{2} + \alpha - 1} e^{-\lambda\left(\frac{1}{2} \sum_{k=1}^M (\ln x_k - \mu)^2 + \beta\right)} d\lambda \\ &= \frac{\left(\frac{M}{2} + \alpha\right)}{\left(\frac{1}{2} \sum_{k=1}^M (\ln \ln x_k - \mu)^2 + \beta\right)} \end{aligned} \tag{45}$$

3.8. Optimal Bonus-Malus System

In the OBMS, the premium calculation was performed using the expected value principle to determine the relative value as written (26), as well as using the comparison formula in (27) for premium determination. Using (40) and (45) obtained earlier, and (26), the equation to calculate the relative value in the OBMS is as follows:

$$\begin{aligned} NR_{n+1} &= E(\theta|k) \times E(\lambda|x) \\ &= \frac{(M+1)(M+2+n+\delta)}{(n+\delta)(M+1+n+\delta)} \times \left(\frac{\left(\frac{M}{2} + \alpha\right)}{\left(\frac{1}{2} \sum_{k=1}^M (\ln \ln x_k - \mu)^2 + \beta\right)} \right). \end{aligned} \tag{46}$$

For the basic relative value when $n = 0$, the values of M , x_k , μ are 0 because no claims have yet occurred, resulting in:

$$NR_1 = \frac{(0+1)(0+2+0+\delta)}{(0+\delta)(0+1+0+\delta)} \times \left(\frac{\left(\frac{0}{2} + \alpha\right)}{\left(\frac{1}{2} \sum_{k=1}^M (\ln 0 - 0)^2 + \beta\right)} \right) = \frac{(2+\delta)}{(\delta)(1+\delta)} \times \left(\frac{\alpha}{\beta} \right). \tag{47}$$

Equation (47) was then used to determine the premiums for subsequent years using (27), producing:

$$Premium_{n+1} = \frac{Premium_1 \times NR_{n+1}}{NR_1} \tag{48}$$

$$= \frac{Premium_1}{\frac{(2 + \delta)}{(\delta)(1 + \delta)} \times \left(\frac{\alpha}{\beta}\right)} \times \frac{(M + 1)(M + 2 + n + \delta)}{(n + \delta)(M + 1 + n + \delta)} \times \left(\frac{\left(\frac{M}{2} + \alpha\right)}{\left(\frac{1}{2} \sum_{k=1}^M (\ln \ln x_k - \mu)^2 + \beta\right)} \right)$$

For example, an insurance company sets the amount of the basic premium to be paid by the policyholder at 1,000,000. If no claim is filed in that year, then using (48), the premium to be paid in the following year is:

$$Premium_{1+1} = \left(\frac{1000}{0.031722}\right) \times \frac{(0 + 1)(0 + 2 + 1 + 2.7106)}{(1 + 2.7106)(0 + 1 + 1 + 2.7106)} \times \left(\frac{\left(\frac{0}{2} + 0.2999\right)}{\left(\frac{1}{2}(0) + 4.4277\right)} \right)$$

The premium amount to be paid in the following year is IDR697,588, which means that the bonus given by the company is: $\frac{1000 - 697.588}{1000} \times 100\% = 30.2412\%$ of the basic premium. If one claim is made in the first year with a claim amount of 500,000, then the premium to be paid in the following year is:

$$Premium_{1+1} = \left(\frac{1000}{0.031722}\right) \times \frac{(1 + 1)(1 + 2 + 1 + 2.7106)}{(1 + 2.7106)(1 + 1 + 1 + 2.7106)} \times \left(\frac{\left(\frac{1}{2} + 0.2999\right)}{\left(\frac{1}{2}(\ln 500 - 10.5574)^2 + 4.4277\right)} \right)$$

The premium to be paid in the following year, if one claim of IDR500,000 occurs, is IDR1,152,529, which means the malus imposed is 15.2529% of the basic premium. However, if two claims occur in the first year with amounts of IDR500,000 and IDR750,000, then the premium to be paid is IDR1,761,655. The malus imposed is 76.1655% of the basic premium. If three claims occur in the first year with amounts of IDR500,000, IDR750,000, and IDR1,000,000, then the premium to be paid in the following year is IDR2,444,306, which means the malus imposed is 144.4306% of the basic premium.

Table 5. Optimal Bonus-Malus System Premiums

		Annual Premium (in thousand rupiah)			
<i>Year</i>	<i>Claim Frequency</i>	<i>0</i>	<i>1</i>	<i>2</i>	<i>3</i>
	<i>Claim Severity (in thousand rupiah)</i>	<i>0</i>	<i>500</i>	<i>750</i>	<i>1,000</i>
0		1,000	1,000	1,000	1,000
1		697.588	1,152.529	1,761.655	2,444.306
2		532.649	887.702	1,364.337	1,900.035
3		429.618	719.937	1,110.592	1,550.693
4		359.448	604.579	935.071	1,308.108
5		308.708	520.590	806.706	1,130.156
6		270.369	456.808	708.887	994.219
7		240.412	406.778	631.950	887.090
8		216.377	366.516	569.899	800.550
9		196.675	333.435	518.826	729.222
10		180.238	305.783	476.071	669.445

Based on this calculation example, a projection table of vehicle insurance premiums can be constructed for $n = 0, 1, 2, \dots, 10$ with a basic premium set at IDR1,000,000, and with several possible claim scenarios: one claim of IDR500,000, two claims of IDR500,000 and IDR750,000, and three claims of IDR500,000, IDR750,000, and IDR1,000,000, as shown in Table 5. Based on Table 5, the amount of vehicle insurance premium refers to the premium amount obtained if, for example, in the first year ($n = 0$) the insured pays a vehicle insurance policy premium of IDR1,000,000. If, during that year, the insured does not make any claims, then in the second year ($n = 1$) the insured only pays IDR697,588. This continues for the following years, so if the insured does not make any claims up to the eleventh year ($n = 11$) they would only pay IDR180,238. Another example: in the fifth year, the insured files three claims, with the first claim amounting to IDR500,000, the second to IDR750,000, and the third to IDR1,000,000. In the previous years, the insured does not made any claims, with the premium payment details as follows: in the second year ($n = 1$) IDR697,588, in the third year ($n = 2$) IDR532,649, in the fourth year ($n = 3$) IDR429,618, and in the fifth year ($n = 4$) IDR 359,448. Therefore, the premium to be paid in the sixth year ($n = 5$) is IDR1,130,156 due to the three claims made in the fifth year with the claim amounts. When no claims are made, the premium paid will continue to decrease. Conversely, when multiple claims are made in one year, particularly with large claim amounts, the premium paid will increase. This occurs due to the application of bonuses and maluses by the insurance company to the insured, in line with the concept of the Bonus–Malus system.

4. Conclusion

The determination of motor vehicle insurance premiums is influenced by historical data on the number of claims and claim amounts. The claim frequency distribution follows the Poisson–Lindley model, while the claim size distribution follows the lognormal–gamma model. Based on these distribution models, the formula for calculating the optimal bonus–malus system premium is obtained, namely multiplying the basic premium by the relative value of the following year divided by the base relative value. The next year's premium value decreases when no claims occur and increase when claims occur. This model provides a more precise and fair premium adjustment mechanism, aligning premiums more closely with individual risk levels. In real insurance practice, the results can be used to improve risk-based pricing, minimize underwriting losses, and promote fairness and transparency in premium determination for policyholders. Future research can explore alternative distributions, such as zero-inflated models for claim frequency and heavy-tailed distributions for claim severity, to better capture extreme values. The use of copula models to account for dependency between frequency and severity can improve aggregate risk estimation. Additionally, integrating machine learning or generalized linear models (GLM) approaches with external risk factors, analyzing temporal trends, and conducting stress testing for extreme scenarios are recommended to enhance predictive accuracy and risk management.

Acknowledgment

The authors would like to thank the Institut Teknologi Sumatera for supporting this research through the Itera Research Grant with contract number 15539ag/IT9.2.1/PT.01.03/2024.

References

- [1] Badan Pusat Statistik, “Jumlah Kendaraan Bermotor Menurut Provinsi dan Jenis Kendaraan (unit), 2023.” [Online]. Available: <https://www.bps.go.id/id/statistics-table/3/VjJ3NGRGa3dkRk5MTIU1bVNFOTVbVbmQyVURSTVFUMDkjMw==/jumlah-kendaraan-bermotor-menurut-provinsi-dan-jenis-kendaraan--unit---2023.html>
- [2] R. Fitriani and Gunardi, “Implementasi metode bayes pada perhitungan premi kendaraan bermotor,” *J. Fundam. Math. Appl. (JFMA)*, vol. 3, no. 2, pp. 112–123, Nov. 2020, doi: 10.14710/jfma.v3i2.8257.
- [3] H. Halim, M. Tumpu, F.E.P. Lapijan, P.R. Rangan, A. Yuniarta, and H. Yusuf, “Patterns and sociodemographic determinants of traffic accidents in Makassar City: A study toward targeted road safety

- interventions,” *Eng. Technol. Appl. Sci. Res.*, vol. 15, no. 6, pp. 29077–29083, Dec. 2025, doi: 10.48084/etasr.13119.
- [4] B.N Rizkia and A.K. Mutaqin, “Bonus-malus premium for third party liability insurance with Poisson-Lindley distribution claim frequency and exponential-inverse gamma distribution claim severity,” *J. Stat.*, vol. 18, no.1, pp. 887–896, Jul. 2025, doi: 10.36456/jstat.vol18.no1.a10327
- [5] I.P. Batubara and R. Syahriza, “Analisis klaim asuransi kendaraan bermotor pada PT Asuransi Jasindo Kantor Cabang Medan,” *J. Soc. Reas.*, vol. 1, no. 9, pp 1026–1035, Sep. 2022, doi: 10.55324/josr.v1i9.62
- [6] E. Stepanes and C.S. Basani, “Legal protection for health insurance policyholders against claim denials due to non-transparent information in insurance policies,” *J. Indones. Sos. Sains*, vol. 7, no. 1, pp 323–335, 2026, doi: 10.59141/jiss.v7i1.2184
- [7] H.A. Shah and A. Ahmed, “Mitigating catastrophic events: Assessing the effectiveness of reinsurance risk management mechanisms,” *Int. J. Acad. Res. Bus. Soc. Sci.*, vol. 14, no. 6, pp. 1558–1563, 2024, doi: 10.6007/IJARBSS/v14-i6/21967.
- [8] L. Girald, “Analysis of factors influencing automobile insurance premiums in France,” *J. Stat. Actuar. Res.*, vol. 8, no. 1, pp. 1 – 10, 2024, doi: 10.47604/jsar.2761.
- [9] R. Oh, J.H.T. Kim, and J.Y. Ahn, “Designing a bonus-malus system reflecting the claim size under the dependent frequency–severity model,” *Probability Eng. Inf. Sci.*, vol. 36, no. 4, pp. 963–987, Oct. 2022, doi: 10.1017/S0269964821000188.
- [10] A. Moumeesri, W. Klongdee, and T. Pongsart, “Bayesian bonus-malus premium with Poisson-Lindley distributed claim frequency and lognormal-gamma distributed claim severity in automobile insurance,” *WSEAS Tran. Math.*, vol. 19, pp. 443–451, 2020, Art. no #46, doi: 10.37394/23206.2020.19.46.
- [11] G.R. Sevina and J. Purwadi, “Bonus malus system for motorized vehicle insurance using geometric distributions and Weibull distributions,” *J. Fundam. Math. Appl.*, vol. 6, no. 1, pp. 71–79, Jun. 2023, doi: 10.14710/jfma.v6i1.16505.
- [12] R.E. Walpole, R.H. Myers, S.L. Myers, and K. Ye, *Probability & Statistics for Engineers & Scientist*. Boston, MA, USA: Perason Education, 2016.
- [13] A. Aijaz, S.Q. Ul Ain, A. Afaq, and R. Tripathi, “Poisson area-biased Ailamujia distribution and its applications in environmental and medical sciences,” *J. Stat. Transit. New Ser.*, vol. 23, no. 3, pp. 167–184, Sep. 2022, doi: 10.2478/stattrans-2022-036.
- [14] M. Shakil, A. Khadim, A. Saghir, M. Ahsanullah, B.M.G. Kibria, and M.I. Bhatti, “Ratio of two independent lindley random variables,” *J. Stat. Theory Appl.*, vol. 21, no. 4, pp. 217–241, Oct. 2022, doi: 10.1007/s44199-022-00050-4
- [15] S.A. Klugman, H.H. Panjer, and G.E. Willmot, *Loss Models: From Data to Decisions*, 5th ed. Hoboken, NJ, USA: Wiley, 2019.
- [16] A. Sutrisno, D. Aziz, A. Hasanah, T. Ruby, and N. Mega, “Comparison of Gauss-Seidel method, Newton-Raphson method, and Broyden method in solving nonlinear equation system,” *Int. J. Adv. Soc. Sci. Edu.*, vol. 3, no. 2, pp. 47–58, Feb. 2025, doi: 10.59890/ijasse.v3i1.292
- [17] K.A. Dalal, et al. “Algorithms of predictor-corrector type with convergence and stability analysis for solving nonlinear systems,” *AIMS Math.*, vol 9, no. 11, pp. 32014–32044, 2024, doi: 10.3934/math.20241538.
- [18] L. Endreyani, *Kinetic Data Analysis*. Toronto, Canada: Springer, 1981.
- [19] T.M. Reshid, “Monte Carlo simulation and derivation of chi-square statistics,” *Am. J. Theor. Appl. Stat.*, vol. 12, no 3, pp. 51–65, 2023, doi: 10.11648/j.ajtas.20231203.13.
- [20] A.A Mir, A.A. Bhat, S.P. Ahmad, B.S. Alnssyan, A. Alsubie, and Y.S. Raghav, “Statistical inference and goodness-of-fit assessment using the AAP-X probability framework with symmetric and asymmetric properties: Applications to medical and reliability data,” *J. Symmetry*, vol. 17, no. 6, 2025, Art. no 863, doi: 10.3390/sym17060863.
- [21] M. Denuit, X. Maréchal, S. Pitrebois, and J.F. Walhin, *Actuarial Modelling of Claim Counts: Risk Classification, Credibility and Bonus-Malus Systems*, Belgium: John Wiley & Sons, Ltd, 2007.