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Generalised difference sequence space of non- absolute type

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Abstract: It was Shiue [16] who have introduced the Cesàro spaces of the type Ces_p and Ces_∞. In view of Chiue, we shall introduce and study some properties of generalised Cesàro difference sequence space. We also examine some of their basic properties viz., BK property and some inclusions relations will be taken care of.

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1 Introduction

By Π we shall denote the set of all sequences (real or complex) and any subspace of it is known as the sequence space. Also, let the set of non-negative integers, the set of real numbers and the set of complex numbers be denoted respectively by $\mathbf{N}$, $\mathbf{R}$ and $\mathbf{C}$. Let l_∞, c and c_0, respectively, denotes the space of all bounded sequences , the space of convergent sequences and the sequences converging to zero. Also, by bs, cs, l_1 and l_p, we denote the spaces of all bounded, convergent, absolutely and p-absolutely convergent series, respectively (see [1-21]).

Suppose $\mathcal{X}$ is a vector space (real or complex) and $H : \mathcal{X} \to \mathbf{R}$. We call $(\mathcal{X}, H)$ a paranormed space with H a paranorm for $\mathcal{X}$ provided :
(i) $H(\theta) = 0$,
(ii) $H(-s) = H(s)$,
(iii) $H(s_1 + s_2) \leq H(s_1) + H(s_2)$, and
(iv) scalar multiplication is continuous, i.e., $|\beta_n - \beta| \to 0$ and $H(s_n - s) \to 0$ gives $H(\beta_n s_n - \beta s) \to 0 \ \forall \ \beta \in \mathbb{R}$ and s's in $\mathcal{X}$, where θ represent zero vector in the space $\mathcal{X}$.

Suppose $\mathcal{A} = (a_{mk})$ be an infinite matrix with $X, Y \subset \Pi$. Then, matrix $\mathcal{A}$ represents the $\mathcal{A}$-transformation from X *into* Y, if for $b = (b_k) \in X$ the sequence $\mathcal{A}b = \{(\mathcal{A}b)_m\}$, the $\mathcal{A}$-transform of b exists and is in Y; where $(\mathcal{A}b)_m = \sum_k a_{mk} b_k$ as can be seen in [24] and many more.

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A FK space $\mathcal{Y}$ is a complete metric sequence space with continuous coordinated $p_m : \mathcal{Y} \to \mathbb{C}$ where $p_m(u) = u_m$ for all $u \in \mathcal{Y}$ and $m \in \mathbb{N}$. A normed FK space is called a BK space as defined in [26] and etc.

Let $\theta = (t_j)$ be increasing integer sequence. Then it will be called lacunary sequence if $t_0 = 0$ and $t_j = t_j - t_{j-1} \to \infty$. By θ we will denote the intervals of the form $I_j = (t_{j-1}, t_j]$ and with q_j we will denote the ratio $\frac{t_j}{t_{j-1}}$ [4].

The spaces $T(\triangle)$ where

$$T(\triangle) = \{u = (u_m) \in \Pi : (\triangle u_m) \in T\}$$

was introduced by Kizmaz [16] where $T \in \{l_\infty, c, c_0\}$ and $\triangle u_m = u_m - u_{m-1}$.

Next Tripathy and Esi [26] had studied it and considered it as follows. Consider the integer $j \geq 0$. then

$$T(\Delta^j) = \left\{ u = (u_k) : \Delta^j u \in T \right\}, \ for\ T = l_\infty, c\ and\ c_0,$$

where $\Delta^j u_i = u_i - u_{i+j}$.

Recently, in [27] we have the following:

$$\Delta_n^m u_k = \{ u \in \Pi : (\Delta_n^m u_k) \in Z\},$$

where

$$\Delta_n^m u_k = \sum_{\mu=0}^{n} (-1)^\mu \binom{n}{r} u_{k+m\mu},$$

and

$$\Delta_n^0 u_k = u_k \forall\ k \in \mathbb{N}.$$

The Cesàro sequence spaces Ces_p and Ces_∞ have been introduced by Shiue [25] and was further studied by several authors viz., Et [3], Orhan[20], Tripathy [27]. Ng and Lee [18] has introduced the Cesàro sequence spaces X_p and X_∞ of non-absolute type and has shown that $Ces_p \subset X_p$ is strict for $1 \leq p \leq \infty$. Our aim in this paper is to bring out the spaces $C_{(p)}\,(\triangle_n^m, \theta)$ and $C_{(p)}\,[\triangle_n^m, \theta]$, where $1 \leq p \leq \infty$ and study their various properties.

2 The spaces $C_{(p)}\,(\triangle_n^m, \theta)$ and $C_{(p)}\,[\triangle_n^m, \theta]$, $(1 \leq p \leq \infty)$.

In this section of text, we introduce the space $C_{(p)}\,(\triangle_n^m, \theta)$ and $C_{(p)}\,[\triangle_n^m, \theta]$, where $1 \leq p \leq \infty$ and prove that these spaces are BK.

Following Başarir [1], Sing [2], Jagers [5], Ganie [6]-[14], Karakaya [15], Mursaleen [17], Nuray [19], Savaş [21]-[23], we introduce for a sequence of strictly positive real numbers $p = (p_i)$, the following spaces:

$$C_{(p)}\left(\triangle_n^m, \theta\right) = \left\{ v = (x_k) : \sum_{i=1}^{\infty} \left| \frac{1}{h_i} \sum_{k \in I_i} \Delta_n^m x_k \right|^{p_i} < \infty \right\},$$

$$C_{(p)}\left[\triangle_n^m, \theta\right] = \left\{ v = (x_k) : \sum_{i=1}^{\infty} \left(\frac{1}{h_i} \sum_{k \in I_i} \Delta_n^m x_k \right)^{p_i} < \infty \right\},$$

$$C_{(\infty)}\left(\triangle_n^m, \theta\right) = \left\{ v = (x_k) : \sup_i \left| \frac{1}{h_i} \sum_{k \in I_i} \Delta_n^m x_k \right|^{p_i} < \infty \right\},$$

and

$$C_{(\infty)}\left[\triangle_n^m, \theta\right] = \left\{ x = (x_k) : \sup_i \frac{1}{h_i} \sum_{k \in I_i} \left|\Delta_n^m x_k\right|^{p_i} < \infty \right\}.$$

It is obvious to see that the above spaces contain some unbounded sequences for $m \geq 1$. To see this, let $\theta = (2^j)$ and $p_j = 1 = n \ \forall j \in \mathbb{N}$, then clearly, $(j^m) \in C_{(\infty)}\left(\triangle_n^m, \theta\right)$ but $(j^m) \notin l_\infty$.

We have the following important result.

Theorem 2.1 *The spaces* $C_{(p)}\left(\triangle_n^m, \theta\right)$, $C_{(p)}\left[\triangle_n^m, \theta\right]$, *are linear spaces.*

Proof : The proof is omitted, as can be proved by special well known techniques.

Theorem 2.2 *For* $1 \leq p < \infty$, *the space* $C_{(p)}\left(\triangle_n^m, \theta\right)$ *is a BK-space normed by*

$$\|x\|_{\Delta_p^\theta} = \sum_{i=1}^{m} |x_i| + \left(\sum_{r=1}^{\infty} \left| \frac{1}{h_r} \sum_{k \in I_r} \Delta_n^m x_k \right|^p \right)^{\frac{1}{p}}$$

and the space $C_{(\infty)}\left(\triangle_n^m, \theta\right)$ *is a BK-space normed by*

$$\|x\|_{\Delta_\infty^\theta} = \sum_{i=1}^{m} |x_i| + \sup_r \left(\left| \frac{1}{h_r} \sum_{k \in I_r} \Delta_n^m x_k \right| \right).$$

Proof : Let $x^j = (x_i^j)_i$ be any Cauchy sequence in $C_{(p)}\left(\triangle_n^m, \theta\right)$ for each $j \in \mathbb{N}$. Therefore, we have

$$\left\|x^i - x^j\right\|_{\Delta_p^\theta} \leq \sum_{t=1}^{m} \left|x_t^i - x_t^j\right| + \sup_r \left(\sum_{r=1}^{\infty} \left| \frac{1}{h_r} \sum_{k \in I_r} \left(\Delta_n^m x_k^i - \Delta_n^m x_k^j\right) \right| \right) \to 0, \text{ as } i, j \to \infty.$$

Hence, $\sum_{t=1}^{m} \left|x_t^i - x_t^j\right| \to 0$ and $\left(\Delta_n^m x_k^i - \Delta_n^m x_k^j\right) \to 0$, as $i, j \to \infty$ for each $k \in \mathbb{N}$. Now, from,

$$\left|x_{t+m}^i - x_{t+m}^j\right| \leq \left|\Delta_n^m x_k^i - \Delta_n^m x_k^j\right| + \binom{m}{0} \left|x_t^i - x_t^j\right| + \cdots + \binom{m}{m-1} \left|x_{m-1}^i - x_{t+m-1}^j\right|,$$

we have $\left|x_t^i - x_t^j\right| \to \infty$ as $i, j \to \infty$, for each $k \in \mathbb{N}$. Therefore, $(x_i^j)_i$ is a Cauchy sequence in $\mathbb{C}$ and hence converges since $\mathbb{C}$ is complete, and let $\lim_i x_t^i = x_t$ for each $t \in \mathbb{N}$. Since x^i is a Cauchy sequence, therefore for each $\epsilon > 0$, we can find $n = n_0(\epsilon)$ such that

$$\left|x^i - x^j\right| < \epsilon \ \forall \ i, j \geq n_0.$$

Thus, we have

$$\lim_j \sum_{t=1}^{m} \left|x_t^i - x_t^j\right| = \sum_{t=1}^{m} \left|x_t^i - x_t\right| < \epsilon$$

and

$$\lim_j \frac{1}{h_r} \sum_{k \in I_r} \left(\Delta_n^m x_k^i - \Delta_n^m x_k^j\right)^p = \frac{1}{h_r} \sum_{k \in I_r} \left(\Delta_n^m x_k^i - \Delta_n^m x_k\right)^p < \epsilon^p$$

for all $r \in \mathbb{N}$ and $i \geq n_0$. This shows that $\|x^i - x\|_{\Delta_p^\theta} < 2\epsilon$, for all $i \geq n_0$. Since,

$$\left|\frac{1}{h_r} \sum_{k \in I_r} \Delta_n^m x_k^i\right|^p \leq 2^p \left(\left|\frac{1}{h_r} \sum_{k \in I_r} \left(\Delta_n^m x_k^{n_0} - x_k\right)\right|^p + \left|\frac{1}{h_r} \sum_{k \in I_r} \Delta_n^m x_k^i\right|^p \right) \to 0$$

as $r \to \infty$, we obtain $x \in C_{(p)}\left(\triangle_n^m, \theta\right)$. Therefore, $C_{(p)}\left(\triangle_n^m, \theta\right)$ is a Banach space. Since, $C_{(p)}\left(\triangle_n^m, \theta\right)$ is a Banach space with continuous co-ordinates, that is, $\|x^i - x\|_{\Delta_p^\theta} \to 0$ for each $k \in \mathbb{N}$ as $i \to \infty$, consequently, it is a BK-space. Hence, the proof of the result is complete.

Theorem 2.3 *$C_{(p)}\left[\triangle_n^m, \theta\right]$ with $1 \leq p < \infty$ is a BK-space with norm*

$$\|x\|_{\Delta_p^\theta} = \sum_{i=1}^{m} |x_i| + \left(\sum_{r=1}^{\infty} \left|\frac{1}{h_r} \sum_{k \in I_r} \Delta_n^m x_k\right|^p \right)^{\frac{1}{p}}$$

and $C_{(\infty)}\left[\triangle_n^m, \theta\right]$ is a BK- space normed by

$$\|x\|_{\Delta_\infty^\theta} = \sum_{i=1}^{m} |x_i| + \sup_r \left(\left|\frac{1}{h_r} \sum_{k \in I_r} \Delta_n^m x_k\right| \right).$$

Proof: The proof is is similar to that of previous theorem and hence can be omitted.

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Theorem 2.4 *The spaces* $C_{(p)}\left[\triangle_n^m,\theta\right]$, $C_{(p)}\left[\triangle_n^m,\theta\right]$, $C_{(\infty)}\left(\triangle_n^m,\theta\right)$, *and* $C_{(\infty)}\left[\triangle_n^m,\theta\right]$ *are neither solid nor symmetric.*

Proof: We only prove the result for $C_{(\infty)}\left[\triangle_n^m,\theta\right]$ and rest can be proven in a similar fashion. So, to establish the result, we put $n = p_j = 1$ for all j and $\theta = (2^r)$. Then, $(u_j) = (j^{m-1}) \in C_{(\infty)}\left[\triangle_n^m,\theta\right]$ but $(\alpha_j u_j) \notin C_{(\infty)}\left[\triangle_n^m,\theta\right]$ where $\alpha_j = (-1)^j$ for all $j \in \mathbb{N}$. Thus, $C_{(\infty)}\left[\triangle_n^m,\theta\right]$ is not solid. This proves the result.

3 Inclusion relations

In this section, we prove some basic inclusion relations for the given spaces.

Theorem 3.1 *For* $m, n \in \mathbb{N}$ *with* $1 \leq p \leq \infty$, *we have*

(i) $C_{(p)}\left(\triangle_n^{m-1},\theta\right) \subset C_{(p)}\left(\triangle_n^m,\theta\right)$,

(ii) $C_{(p)}\left[\triangle_n^{m-1},\theta\right] \subset C_{(p)}\left[\triangle_n^m,\theta\right]$,

(iii) $C_{(p)}\left[\triangle_n^m,\theta\right] \subset C_{(q)}\left[\triangle_n^m,\theta\right]$,

(iv) $C_{(p)}\left(\triangle_n^m,\theta\right) \subset C_{(q)}\left(\triangle_n^m,\theta\right)$ *where* $0 < p < q < \infty$.

Proof : We shall only prove (i). So, let $x \in C_{(p)}\left(\triangle_n^{m-1},\theta\right)$. Then, we have

$$\left|\frac{1}{h_r}\sum_{k\in I_r}\Delta_n^m x_k\right| \leq \left|\frac{1}{h_r}\sum_{k\in I_r}\Delta_n^{m-1} x_k\right| + \left|\frac{1}{h_r}\sum_{k\in I_r}\Delta_n^{m-1} x_{k+1}\right|.$$

Hence,

$$\left|\frac{1}{h_r}\sum_{k\in I_r}\Delta_n^m x_k\right|^p \leq 2^p\left(\left|\frac{1}{h_r}\sum_{k\in I_r}\Delta_n^{m-1} x_k\right| + \left|\frac{1}{h_r}\sum_{k\in I_r}\Delta_n^{m-1} x_{k+1}\right|^p\right).$$

Thus, for each positive integer r_0, we have

$$\sum_{r=1}^{r_0}\left|\frac{1}{h_r}\sum_{k\in I_r}\Delta_n^m x_k\right|^p \leq 2^p\left(\sum_{r=1}^{r_0}\left|\frac{1}{h_r}\sum_{k\in I_r}\Delta_n^{m-1} x_k\right| + \left|\sum_{r=1}^{r_0}\frac{1}{h_r}\sum_{k\in I_r}\Delta_n^{m-1} x_{k+1}\right|^p\right).$$

Now, taking $r_0 \to \infty$ in the above inequality, we see that

$$\sum_{r=1}^{\infty}\left|\frac{1}{h_r}\sum_{k\in I_r}\Delta_n^m x_k\right|^p \leq 2^p\left(\sum_{r=1}^{\infty}\left|\frac{1}{h_r}\sum_{k\in I_r}\Delta_n^{m-1} x_k\right| + \left|\sum_{r=1}^{\infty}\frac{1}{h_r}\sum_{k\in I_r}\Delta_n^{m-1} x_{k+1}\right|^p\right).$$

Consequently, $C_{(p)}\left(\triangle_n^{m-1},\theta\right) \subset C_{(p)}\left(\triangle_n^m,\theta\right)$.

To show inclusion is proper, we see that the sequence $x = (k^{m-1})$ belongs to $C_{(p)}(\triangle_n^m, \theta)$ but does not belong to $C_{(p)}(\triangle_n^{m-1}, \theta)$, for $\theta = (2^r)$. This completes the proof.

References

[1] M. Başarir, Paranormed Cesàro difference sequence space and related matrix transformation, Turk. J. Math., 15(1991), 14-19.

[2] V. K. Bhardwaj and N. Sing, On some new spaces of lacunary strongly σ-convergent sequences defined by Orlicz functions, Indian J. Pure Appl. Math., 31(11)(2000), 1515-1526.

[3] M. Et, On some generalised Cesàro difference sequence spaces, Istanbul Univ. Fen Fak. Mat. Derg., 55(56)(1996/97), 221-229.

[4] A. R. Freedman, J. J. Sember and M. Raphael, Some Cesàro type summability spaces, Proc. Lond. Math. Soc., 37(1978), 508-520.

[5] A. A. Jagers, A note on Cesàro sequence spaces, Nieuw. Arch. Wisk., 22(3)(1974), 113-124.

[6] A. H. Ganie and A. Antesar, Certain sequence spaces, Adv. Stu. in Comtem. Math., 30(1)(2020), 17-27.

[7] A. H. Ganie and N. A. Sheikh, Matrix transformation into a new sequence space related to invariant means, Chamchuri J. Math., 4(12)(2012), 71-72.

[8] A. H. Ganie and N. A. Sheikh, On some new sequence space of non-absolute type and matrix transformations, Jour. Egyptain Math. Soc., 21(2013), 34-40.

[9] A. H. Ganie and N. A. Sheikh, Infinite matrices and almost bounded sequences, Vietnam J. Math, 42(2)(2014), 153-157.

[10] A. H. Ganie, A. Mobin N. A. Sheikh and T. Jalal, New type of Riesz sequence space of non-absolute type, J. Appl. Comput. Math., 5(1)(2016), 1-4.

[11] A. H. Ganie, N. A. Sheikh, M. Ahmad and Afroza, Some matrix mappings of σ-means, Asian Journal of Mathematics and Computer Research,13(2)(2016), 119-124.

[12] A. H. Ganie and N. A. Sheikh, Infinite matrices and almost convergence, Filomat, 29(6) (2015), 1183-1188.

[13] A. H. Ganie, B. C. Tripathy, N. A. Sheikh and M. Sen, Invariant means and matrix transformations, Func. Anal.-TMA, 2 (2016), 28-33.

[14] T. Jalal, S. A. Gupkari and A. H. Ganie, Infinite matrices and sigma convergent sequences, Southeast Asian Bull. Math., 36 (2012), 825-830.

[15] V. Karakaya, H. Kara and R. Savaş, On strongly Crsàro convergent sequence spaces defined by Orlicz function, Analele Universităţii din Timi¸soara, XL, (2)(2002), 67-74.

[16] H. Kizmaz, On certain sequence, Canad, Math. Bull., 24(2) (1981), 169-176.

[17] M. Mursaleen, A. H. Ganie and N. A. Sheikh, New type of generalized difference sequence space of non-absolute type and some matrix transformations, Filomat, 28(7)2014), 1381-1392.

[18] P.-N, Ng and P.-Y.Lee, Cesàro sequences spaces of non-absolute type, Comment. Math. Prace Mat., 20(2)(1978), 429-433.

[19] F. Nuray and E.Savaş, Some new sequence spaces defined by a modulus function, Indian J. Pure Appl. Math., 24 (4)(1993), 657-663.

[20] C. Orhan, Cesàro difference sequence spaces related matrix transformations, Comm. Fac. Sci. Univ. Ankara Ser. A1 Math., 32(8)(1983), 55-63.

[21] E. Savaş, Spaces of strongly lacunary invariant summable sequences, Tbilisi Math. J., 13(1)(2020), 61-68.

[22] E. Savaş and R. Savaş, Some sequence spaces defined by Orlicz function, Arch. Math. (Brno), 40(2004), 33-40.

[23] E. Savaş, Lacunary almost convergence and some new sequence spaces, Filomat, 33(5)(2019), 1397-1402.

[24] N. A. Sheikh and A. H. Ganie, A new paranormed sequence space and some matrix transformation, Acta Math. Academiae Paedag. Nyiregy., 28 (2012), 47–58.

[25] J. S. Shiue, On the Cesàro sequence space, Tamkang J. Math., 2(1970), 19-25.

[26] B. C. Tripathy and A. Esi, A new type of difference sequence spaces, Int. J. Sci. Tech., 1(1)(2006), 11-14.

[27] B. C. Tripathy, A. Esi and B. K. Tripathy , On a new type of generalized difference Cesàro sequence spaces, Soochow J. Math., 31(3)(2005), 333-340.