Research Article

Lie Symmetry and Lie Bracket in Solving Differential Equation Models of Functional Materials: A Survey

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Abstract: Functional materials are becoming an increasingly important part of our daily life, e.g. they used for sensing, actuation, computing, energy conversion. These materials often have unique physical, chemical, and structural characteristic involving very complex phase. Many mathematical models have been devised to study the complex behavior of functional materials. Some of the models have been proven powerful in predicting the behavior of new materials built upon the composites of existing materials. One of mathematical methods used to model the behavior of the materials is the differential equation. Very often the resulting differential equations are very complicated so that most methods failed in obtaining the exact solutions of the problems. Fortunately, a relatively new approach via Lie symmetry gives a new hope in obtaining or at least understanding the behavior of the solutions, which is needed to understand the behavior of the materials being modeled. In this paper we present a survey on the use of Lie symmetry and related concepts (such as Lie algebra, lie group, etc) in modeling the behavior of functional materials and discuss some fundamental results of the Lie symmetry theory which often used in solving differential equations. The survey shows that the use of Lie symmetry and alike have been accepted in many fields and gives an alternative approach in studying the complex behavior of functional materials.

Keywords: mathematical model of functional material, differential equation, vector field, Lie symmetry

Introduction

The fundamental ideas of Lie algebra was introduced by Sophus Lie more than century ago (1842-1899). He studied actions of groups on manifolds and these actions were investigated infinitesimally. Furthermore, Lie algebras correspond to Lie groups notion. Many aspects and types of Lie algebras and Lie groups attracted to study. A semi simple Lie algebra is the most interesting ones [1]. Another type is Frobenius Lie algebras which is introduced in the first time in [2]. Particularly, harmonic analysis for 4-dimensional real Frobenius Lie algebras can be found in [3] which also studied representations of 4-dimensional real Frobenius Lie groups. Roughy speaking, a representation can be thought as an action of a group on a certain carrier space.

In the context of differential equations, representations theory of Lie groups is very important. To see this, let us assume that rotational symmetry is contained in a three-dimensional space of differential equation. This implies that the space of solution is invariant. This means that a representations of the rotation group SO(3) can be applied to the space of solutions(see [4] for detail) . In other words, representation theory can be useful to understand symmetry of differential equations. Indeed, the solutions set of a differential equations which symmetry is contained shall form a group and this can simplify the problems of differential equations.

On the other hand, in the application perspective, mathematical models have been used in studying functional materials for long time. The models are usually directed to understand the properties of materials, either solids, liquids, amorphous, and their interface. Two main purposes in this area are to predict the behavior of materials and to design functional materials [5]. Different mathematical methods are possible to be used, such as response function of matrix scheduling [6], differential equation and optimization [7], finite element method [8], geometric representation [9], matrix and probability [10], symmetry methods [11], Lie grupoid [12], symmetry method [14], and [13].

In the case of the models appeared as differential equations, some attempt to find exact solutions are done through the use of symmetry, Lie group and Lie bracket ([15], [16], [17], and [18]). In the following sections we discuss some algebraic aspects related to symmetry and Lie group, and Lie bracket related to the method in solving differential equations.

PRELIMINARIES

In this section, we briefly review some basic notions of Lie groups, Lie algebras, Lie symmetries, and some aspects of them related to the method in determining the solutions of differential equations.

A LIE GROUP AND A LIE ALGEBRA

Definition 2.1. [5, p. 286]. Let G be a group. G is said to be a Lie group if G is equipped with the structure of a smooth manifold such that the following operations are smooth.

1. $\times : G \times G \ni (\alpha, \beta) \mapsto \times (\alpha, \beta) = \alpha\beta \in G$.
2. $I : G \ni \alpha \mapsto I(\alpha) = \alpha^{-1} \in G, \quad (\forall \, \alpha, \beta \in)$.

Example 2.2. Let $G := \mathbb{R}^n$ be a additive group equipped with the natural smooth manifold. Indeed, the smooth structure on G given by the atlas of the chart (Id, G). Thus, the following operations

$$\mathbb{R}^n \times \mathbb{R}^n \ni (\alpha, \beta) \mapsto \alpha + \beta \in \mathbb{R}^n \quad \text{and} \quad \mathbb{R}^n \ni \alpha \mapsto -\alpha \in \mathbb{R}^n. \tag{1}$$

are smooth.

Furthermore, we discuss the notion of the Lie bracket, which is a way of operations of smooth vector fields in order to obtain new one. We notice here that the set of smooth left-invariant vector fields on a Lie group G forms what we called a Lie algebra of a Lie group (see [5] and [6] for more details).

Definition 2.3. [6, p. 174]. Let M be a smooth manifold and $TM = \prod_{\zeta \in M} T_\zeta M$ be the tangent bundle of M. A vector field on M is a continuous map

$$X : M \ni \zeta \mapsto X(\zeta) := X_\zeta \in T_\zeta M. \tag{2}$$

We are interested in smooth vector fields, namely, the map $X : M \to TM$ is smooth. In this case, the tangent bundle TM can be considered as the smooth manifold.

Definition 2.4. [6, p. 186]. Let X and Y be two smooth vector fields on a smooth manifold M. The Lie bracket of X and Y is a map given by

$$[X, Y] : C^\infty(M) \ni \psi \mapsto [X, Y]\psi \in C^\infty(M). \tag{3}$$

and defined by

$$[X, Y]\psi = XY\psi - YX\psi \tag{4}$$

where $C^\infty(M)$ is set of smooth fuctions on M.

Remark 2.5. A Lie bracket of smooth vector fields is a smooth vector field.

Example 2.6. Here some examples of Lie bracket computations

1. Let X_1 and X_2 be two smooth vector fields on $\mathbb{R}^3$ and defined by

$$X_1 = x_2 \frac{\partial}{\partial x_3} - 2x_1 x_2^2 \frac{\partial}{\partial x_2},$$

$$X_2 = \frac{\partial}{\partial x_2}.$$

Then we compute that $[X_1, X_2] = -\frac{\partial}{\partial x_3} + 4x_1 x_2 \frac{\partial}{\partial x_2}$.

2. Let $X_1 = -x_3 \frac{\partial}{\partial x_2}$ and $X_2 = x_3 \frac{\partial}{\partial x_1}$ be vector fields on $\mathbb{R}^3$, then $[X_1, X_2] = 0$.

Let G be a Lie group and X be a smooth vector field on G. We recall a left translation as the action of G on itself and for $x \in G$, the translation is given by $\varrho_x : G \ni g \mapsto \varrho_x(g) := xg \in G$. We are coming to the notion of left-invariant vector field on G.

Definition 2.7. [5, p. 189]. Let X be a vector fields on a Lie group G. X is said to be left-invariant if

$$(\varrho_x)_* X = X, \qquad (5)$$

For every $x \in G$.

Remark 2.8. If X and Y are smooth left-invariant vector fields, so is $[X, Y]$.

Furthermore, We are ready to define the notion of the Lie algebra $\mathfrak{g}$ of the Lie group G which $\mathfrak{g}$ can be considered as the set of smooth left-invariant vector fields on G.

Definition 2.9. [7, p. 1]. A real vector space $\mathfrak{g}$ is said to be a Lie algebra if $\mathfrak{g}$ is edndowed with a bracket given by

$$[\,,] : \mathfrak{g} \times \mathfrak{g} \ni (A, B) \mapsto [A. B], \qquad (6)$$

satisfying the following properties

1. The bracket $[\,,]$ is bilinear, that is, for $\alpha, \beta \in \mathbb{R}$

$$[\alpha A + \beta B, C] = \alpha[A, C] + \beta[B, C] \text{ and } [C, \alpha A + \beta B] = \alpha[C, A] + \beta[C, B]. \qquad (7)$$

2. The bracket $[\,,]$ is antysimmetry.
$$[A, B] = -[B, A]. \qquad (8)$$

3. The bracket $[\,,]$ satisfies the Jacoby identity.

$$[[A, B], C] = [A, [B, C]] + [B, [C, A]]. \qquad (9)$$

for all $A, B, C \in \mathfrak{g}$.

Example 2.10. The following are familiar examples of Lie algebras

1. The space $M_n(\mathbb{R})$ consisting of $n \times n$ real matrices is a Lie algebra with bracket $[A, B] = AB - BA$.
2. Let M be a smooth manifold and $\Gamma(M)$ be set of smooth vector fields on M, then $\Gamma(M)$ is a Lie algebra. Furthermore, $\Gamma^l(M) \subset \Gamma(M)$ consisting of left-invariant smooth vector fields on M is also a Lie algebra.

One of important things is about one-parameter Lie group as can be seen as follows.

Theorem 2.11. [5, p.305]. Let G be a Lie group whose Lie algebra is $\mathfrak{g}$. Each continuous one-parameter group

$$\xi : \mathbb{R} \supseteq I \to G, \qquad\qquad (11)$$

can be written as a smooth group homomorphism $\xi_a : \mathbb{R} \supseteq I \ni t \mapsto \exp(ta) \in G$ for every $a \in \mathfrak{g}$.

Lie Symmetries Of Ordinary Differential Equations (ODE)

To understand the notion of symmetries of differential equation, we interpret this notion in geometrical object as explained in [13, 14, 15]. This means, a symmetry is a transformation that preserves the structure of object. Let us imagine that an equilateral triangle was rotated 120° through its center. We can see that this unchanged triangle. This idea is generalized to the case of symmetries of differential equations (see [14] for more examples). Roughly speaking, a symmetry is a transformation given by

$$\psi : y \mapsto \hat{y} \qquad\qquad (12)$$

Where y and $\hat{y}$ are both solutions of a given differential equation.

Let $x' = \frac{dx}{dt} = \pi(t, x)$ be a first-order of ordinary differential equation (linear or non-linear) and Δ be the set of solution in (x, t) form. We must find a symmetry that maps such solution to the same set of solution $\bar{\Delta}$ in $(\bar{t}, \bar{x})$ form. In other words, we should find that

$$\psi : (t, x) \mapsto (\bar{t}, \bar{x}). \qquad\qquad (13)$$

is a symmetry. Thus, the formula

$$\bar{x}' = \frac{\partial_t(\bar{x}) + x'\partial_x(\bar{x}) + x''\partial_{x'}(\bar{x}) + \cdots}{\partial_t(\bar{t}) + x'\partial_x(\bar{t}) + x''\partial_{x'}(\bar{t}) + \cdots} = \pi(\bar{t}, \bar{x}) \ \text{ when } x' = \pi(t, x). \qquad (14)$$

is the symmetry condition for x', where $\partial_x = \frac{\partial}{\partial x}$.

Example 2.12. [14, p. 13]. For the differential equation

$$x' = \frac{dx}{dt} = x^2 e^{-t} + x + e^x,$$

$\Gamma_\varepsilon : (t, x) \mapsto (\bar{t}, \bar{x}) = (t + 2\varepsilon, xe^{2\varepsilon})$ is a symmetry. Indeed, using (14) we obtain that $\bar{x}' = x'e^{2\varepsilon}$ when $x' = x^2 e^{-t} + x + e^x$.

■

A one-parameter Lie group can act on points in the plane. By a suitable choice ε we can map (t, x) and we can collect these points in what we called the orbit Θ at the point (t, x). By choosing $\varepsilon = 0$ we have $(\bar{t}, \bar{x}) = (t, x)$. The points are called invariants if they can be mapped to itself by using the Lie symmetries. Let $\alpha(\bar{t}, \bar{x})$ and $\beta(\bar{t}, \bar{x})$ be the tangent vector to the orbit Θ at the point $(\bar{t}, \bar{x})$, where

$$\alpha(\bar{t}, \bar{x}) = \frac{d\bar{t}}{d\varepsilon} \qquad \text{and} \qquad \beta(\bar{t}, \bar{x}) = \frac{d\bar{x}}{d\varepsilon}. \qquad\qquad (15)$$

Particularly, for $\varepsilon = 0$ in the latter formulas, we have $\alpha(t, x)$ and $\beta(t, x)$. In the **Example 2.12.**, we can see that

$$\big(\alpha(t, x), \beta(t, x)\big) = (2, 2x).$$

Now we shall introduce the notion of canonical coordinates. We introduce the coordinates

$$(p, q) = (p(t, x), q(t, x)) \text{ such that } (\bar{p}, \bar{q}) = (p, q + \varepsilon). \qquad (16)$$

We can obtain these canonical coordinates by using characteristic equations. Let $f(t, x)$ be a solution for the equation

$$\frac{dx}{dt} = \frac{\beta(t,x)}{\alpha(t,x)} \tag{17}$$

We obtain $p(t, x) = f(t, x)$. The coordinate $q(t, x)$ can be computed by solving

$$\left(\int \frac{1}{\alpha(t,x(p,t))} dt \right) \qquad \text{where } p = p(t,x) \tag{18}$$

Solution for First Order of ODE Using Lie Symmetries

To make clear our discussion before, let us give the complete computations for the Riccati equation in [14, p.27] as follows:

Example 2.13. [14, p. 27]. Let the differential equation

$$x' = \frac{dx}{dt} = tx^2 - \frac{2x}{t} - \frac{1}{t^3}, \qquad (t > 0) \tag{19}$$

be the first order ODE. Using (14) we can compute that $\bar{x}' = \frac{d\bar{x}}{d\bar{t}} = \frac{x'}{e^{3\varepsilon}} = \pi(\bar{t}, \bar{x})$ when $\pi(t, x) = tx^2 - \frac{2x}{t} - \frac{1}{t^3}$, where $(\bar{t}, \bar{x}) = (e^{\varepsilon}x, e^{-2\varepsilon}y)$. Thus, $(\bar{t}, \bar{x}) = (e^{\varepsilon}x, e^{-2\varepsilon}y)$ is one of Lie symmetries for this ODE. Moreover, the tangent vectors at $\varepsilon = 0$ is of the forms $\alpha(t, x) = t$ and $\beta(t, x) = -2x$. Using formulas (17) and (18) we obtain the canonical coordinates as follow:
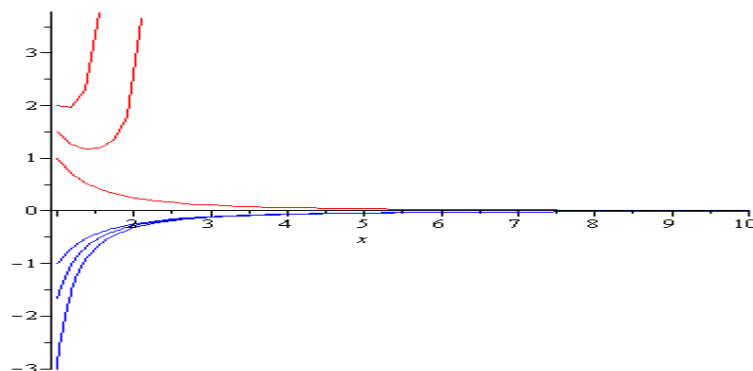
1. $\int \frac{1}{x} dx = \int \frac{-2}{t} dt$. Thus we get $p(t, x) = t^2 x$.
2. $q(t, x) = \int \frac{1}{t} dt = \ln ct$. In other words, $(p(t, x), q(t, x)) = (t^2 x, \ln t)$.

Applying chain rules, we can compute that

$$\frac{dq}{dp} = \frac{\frac{\partial q}{\partial t} + \pi(t,x)\frac{\partial q}{\partial x}}{\frac{\partial p}{\partial t} + \pi(t,x)\frac{\partial p}{\partial x}} = \frac{\frac{1}{t}}{2tx + \pi(t,x)t^2} = \frac{1}{t^4 x^2 - 1} = \frac{1}{p^2 - 1} \; .$$

Solving the latter equation we obtain $q = \frac{1}{2} \ln \left| \frac{p-1}{p+1} \right|$. Therefore, the solution is of the form $x = \frac{t^2 + k}{-t^4 + kt^2}$ where k is a constant.

Further information about this solution of (19) can be seen in Figure 1. below

Figure 1. Plots of the solutions from a numerical scheme for various initial values at t=1 (red) and from the Lie Symmetries for various values of k (blue).

Conclusions

We discussed some fundamental results of the Lie symmetry theory which effectively used in solving differential equations. Lie symmetry gives a new tool to find exact solutions for differential equations. Although in this survey we use the Lie symmetry for the first-order differential equations but in fact the method can be generalized for higher order ODEs and system of ODEs.

Acknowledgements

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